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The hydrodynamic form of the Dirac equation and the distorted wave Glauber approximation

Frank Riordan

Mathematics Department, University of Birmingham, B15 2TT, England

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Abstract. The Dirac equation for an electron subjected to a potential A^μ is shown to be equivalent to the following classical equations for a beam of non-interacting electrons subjected to A^μ in addition to 'quantum' scalar and four-vector potentials together with an anomalous electromagnetic field affecting the spin: the Hamilton–Jacobi equation for the momentum, the continuity equation for the charge distribution, Frenkel's relativistic equation for the spin and a new equation giving a phase θ as an integral along the electron's path. These equations are coupled through the 'quantum' potentials and fields. In the classical limit θ is constant along the electron's path and the equations decouple.

Two approximation procedures based on these coupled hydrodynamic equations are given: (1) the semiclassical to lowest order in powers of $\hbar$, (2) the distorted wave Glauber approximation assumes the solution of the Dirac equation and hence of the hydrodynamic equations for an unperturbed A^μ and that the electrons travel along the same four-dimensional paths in the perturbed potential, but with different 'speed'.

1. Introduction

It is well known that the Hamiltonian Jacobi (HJ) equation

$$\nabla S \cdot \nabla S + 2m\tilde{V} = 2Em \tag{1.1}$$

describing the velocity field $(1/m)\nabla S(\boldsymbol{x})$ of particles of fluid subjected to a potential $\tilde{V}$, together with the continuity equation (CE)

$$\nabla \cdot (a^2 \nabla S) = 0 \tag{1.2}$$

governing the density $a^2(\boldsymbol{x})$ of such particles moving with velocity field $(1/m)\nabla S$, are equivalent to the Schrödinger equation with potential $V(\boldsymbol{x})$ for the wavefunction

$$\psi(\boldsymbol{x}) = a(\boldsymbol{x}) \exp[\mathrm{i}S(\boldsymbol{x})/\hbar] \tag{1.3}$$

if we take

$$\tilde{V}(\boldsymbol{x}) - V(\boldsymbol{x}) = -\hbar^2\nabla^2 a(\boldsymbol{x})/2ma(\boldsymbol{x}). \tag{1.4}$$

The dependence of the 'quantum potential'

$$V_{\mathrm{q}}(\boldsymbol{x}) = -\hbar^2\nabla^2 a(\boldsymbol{x})/2ma(\boldsymbol{x}) \tag{1.5}$$

on the amplitude $a(\boldsymbol{x})$ couples the CE (1.2) and the HJ equation (1.1).

The Pauli equation for the non-relativistic electron requires (Riordan 1978) 'quantum vector potentials' and 'quantum magnetic fields' if it is to be written as a HJ

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equation governing the velocity field of the electrons together with the CE governing the density distribution of electrons and the spin equation

$$DM = (e/mc)M \times \tilde{H} \tag{1.6}$$

(where D denotes the substantive derivative). This equation describes the precession of the magnetic moment M of the electron about the direction of the magnetic field $\tilde{H}$ (classical plus quantum) which the electron finds at each position as it moves through space with velocity field given by the HJ equation.

In this paper we write down the relativistic extension of these hydrodynamic equations and show that they are equivalent to the Dirac equation in a classical external potential field (ϕ, A). The HJ equation is the relativistic Hamilton equation for the four-momentum $\partial_\mu S$ in the presence of the classical and 'quantum' four-vector potential A_μ and a 'quantum' scalar potential. The CE is as we would expect for such a four-momentum field. The remaining equation for the magnetic and electric moment densities $(-M, P)$, which like the magnetic and electric fields (H, E) form a covariant six-vector (antisymmetric tensor of rank two), is just Frenkel's (1926) relativistic equation which is equivalent to the more usual equations of Bargmann *et al* (1959) for the four-vector a^μ derived from the spin and the kinetic momentum $p = \nabla S - (e/c)\tilde{A}$, as follows:

$$a^0 = (M \cdot p)/E, \qquad a = M + p(M \cdot p)/m(E+m). \tag{1.7}$$

2. The hydrodynamic equations

The Dirac equation in the presence of a vector potential ϕ, A may be written in the spinor representation

$$\psi = a \exp(\mathrm{i}S/\hbar)\begin{pmatrix}\xi \\ \eta\end{pmatrix} \tag{2.1}$$

as

$$\begin{pmatrix} -m & \mathrm{i}\hbar\partial - \phi + (\mathrm{i}\hbar\nabla + A)\cdot\sigma \\ \mathrm{i}\hbar\partial - \phi - (\mathrm{i}\hbar\nabla + A)\cdot\sigma & -m \end{pmatrix} a\, \mathrm{e}^{\mathrm{i}S/\hbar}\begin{pmatrix}\eta \\ \xi\end{pmatrix} = 0 \tag{2.2}$$

where

$$\xi = \begin{pmatrix}\xi^1 \\ \xi^2\end{pmatrix}, \qquad \eta = \begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix}, \qquad \xi^* = \begin{pmatrix}\xi^{1*} \\ \xi^{2*}\end{pmatrix} \tag{2.3}$$

are two-spinors. η transforms like ξ^* and we may define

$$\xi_1 = \xi^2, \qquad \xi_2 = -\xi^1, \qquad \omega = \eta^\dagger\xi + \xi^\dagger\eta. \tag{2.4}$$

$a^2(x)\omega(x)$ is the Lorentz scalar.

$a(x)$ may be made definite by for example setting $\omega = 1$; $S(x)$ and $a(x)$ could be chosen to be solutions of the classical HJ and continuity equations respectively if we know these; alternatively both $S(x)$ and $a(x)$ can also be made definite by setting $\xi^1 = 1$, but we will leave this choice open until §§ 3 and 4. The Dirac equation (2.2) may be rewritten

$$ma\eta = (\mathrm{i}\hbar\partial - \partial S - \phi)a\xi + (\mathrm{i}\hbar\nabla - \nabla S + A)a \cdot \sigma\xi, \tag{2.5a}$$

$$ma\xi = (\mathrm{i}\hbar\partial - \partial S - \phi)a\eta - (\mathrm{i}\hbar\nabla - \nabla S + A)a \cdot \sigma\eta, \tag{2.5b}$$

so that the density

$$\rho = a^2(\xi^\dagger\xi + \eta^\dagger\eta) \tag{2.6}$$

is given by

$$m\rho = \xi^\dagger a[\mathrm{i}\hbar\partial - \partial S - \phi - \boldsymbol{\sigma}\cdot(\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A})]a\eta$$
$$+\eta^\dagger a[\mathrm{i}\hbar\partial - \partial S - \phi + \boldsymbol{\sigma}\cdot(\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A}]a\xi \tag{2.7}$$
$$= -a^2\omega(\partial S + \tilde{\phi}) - \boldsymbol{\nabla}\cdot[\tfrac{1}{2}a^2\mathrm{i}\hbar(\xi^\dagger\boldsymbol{\sigma}\eta - \eta^\dagger\boldsymbol{\sigma}\xi)] \tag{2.8}$$

since the imaginary part of the right-hand side of equation (2.7) is zero and we have written

$$\tilde{\phi} - \phi \equiv \phi_{\mathrm{q}} \equiv \hbar\,\mathrm{Im}(\xi^\dagger\partial\eta + \eta^\dagger\partial\xi)/\omega. \tag{2.9}$$

Similarly writing

$$\tilde{\boldsymbol{A}} - \boldsymbol{A} \equiv \boldsymbol{A}_{\mathrm{q}} \equiv -\hbar\,\mathrm{Im}(\xi^\dagger\boldsymbol{\nabla}\eta + \eta^\dagger\boldsymbol{\nabla}\xi)/\omega \tag{2.10}$$

and using

$$(\boldsymbol{B}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} = \boldsymbol{B} - \mathrm{i}\boldsymbol{B}\times\boldsymbol{\sigma}, \tag{2.11}$$

the current

$$\boldsymbol{j} = a^2(\xi^\dagger\boldsymbol{\sigma}\xi - \eta^\dagger\boldsymbol{\sigma}\eta) \tag{2.12}$$

is given by

$$m\boldsymbol{j} = \xi^\dagger a[(\mathrm{i}\hbar\partial - \partial S - \phi)\boldsymbol{\sigma}a\eta - (\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A})a\eta - \mathrm{i}(\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A})\times\boldsymbol{\sigma}a\eta]$$
$$+\eta^\dagger a[-(\mathrm{i}\hbar\partial - \partial S - \phi)\boldsymbol{\sigma}a\xi - (\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A})a\xi$$
$$-\mathrm{i}(\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A})\times\boldsymbol{\sigma}a\xi] \tag{2.13}$$
$$= a^2\omega(\boldsymbol{\nabla}S - \tilde{\boldsymbol{A}}) + \tfrac{1}{2}\mathrm{i}\hbar\partial a^2(\xi^\dagger\boldsymbol{\sigma}\eta - \eta^\dagger\boldsymbol{\sigma}\xi) + \tfrac{1}{2}\hbar\boldsymbol{\nabla}\times a^2(\xi^\dagger\boldsymbol{\sigma}\eta + \eta^\dagger\boldsymbol{\sigma}\xi). \tag{2.14}$$

That $(\rho, \boldsymbol{j})$ is a four-vector follows from the definitions (2.6) and (2.12). Multiplying equation (2.2) on the left by $(\xi^\dagger, \eta^\dagger)$ and keeping only the imaginary part, we get the continuity equation

$$\partial[a^2(\xi^\dagger\xi + \eta^\dagger\eta)] + \boldsymbol{\nabla}\cdot[a^2(\xi^\dagger\boldsymbol{\sigma}\xi - \eta^\dagger\boldsymbol{\sigma}\eta)] = 0, \tag{2.15}$$

or using equations (2.8) and (2.14)

$$\partial[a^2\omega(\partial S + \tilde{\phi})] - \boldsymbol{\nabla}[a^2\omega(\boldsymbol{\nabla}S - \tilde{\boldsymbol{A}})] = 0. \tag{2.16}$$

The Dirac equation (2.5) yields on iteration the second-order equation

$$[(\mathrm{i}\hbar\partial - \partial S - \phi)(\mathrm{i}\hbar\partial - \partial S - \phi) - (\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A})\cdot\boldsymbol{\sigma}\,\boldsymbol{\sigma}\cdot(\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A}) - m^2]a\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$
$$+[(\mathrm{i}\hbar\partial - \partial S - \phi)(\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A})\cdot - (\mathrm{i}\hbar\boldsymbol{\nabla} - \boldsymbol{\nabla}S + \boldsymbol{A})\cdot(\mathrm{i}\hbar\partial - \partial S - \phi)]a\begin{pmatrix}\boldsymbol{\sigma}\xi\\ -\boldsymbol{\sigma}\eta\end{pmatrix} = 0 \tag{2.17}$$

or

$$a[(\partial S+\tilde{\phi})^2-(\nabla S-\tilde{A})^2-m^2-\phi_q^2+A_q^2]\binom{\xi}{\eta}+\hbar a\binom{(H-\mathrm{i}E)\cdot\sigma\xi}{(H+\mathrm{i}E)\cdot\sigma\eta}$$

$$-\hbar^2(\partial^2-\nabla^2)a\binom{\xi}{\eta}-\frac{\mathrm{i}\hbar}{a}\{\partial[a^2(\partial S+\phi)]-\nabla[a^2(\nabla S-A)]\}\binom{\xi}{\eta}$$

$$-2a[\phi_q(\partial S+\phi)+A_q\cdot(\nabla S-A)]\binom{\xi}{\eta}$$

$$-2\mathrm{i}\hbar a[(\partial S+\phi)\partial-(\nabla S-A)\cdot\nabla]\binom{\xi}{\eta}=0 \tag{2.18}$$

where we have used (2.11) and

$$H\xi=\nabla\times(A\xi)+A\times\nabla\xi, \tag{2.19}$$

$$E\xi=-\partial(A\xi)+A\partial\xi-\nabla(\phi\xi)+\phi\nabla\xi. \tag{2.20}$$

If we write the substantive derivative

$$D\equiv(\partial S+\phi)\partial-(\nabla S-A)\cdot\nabla \tag{2.21}$$

the term in braces in (2.18) is given, using the continuity equation (2.16), by

$$\omega\{\partial[a^2(\partial S+\phi)]-\nabla\cdot[a^2(\nabla S-A)]\}$$

$$=-a^2D\omega-\partial(a^2\omega\phi_q)-\nabla\cdot(a^2\omega A_q) \tag{2.22}$$

$$=-2a^2\,\mathrm{Re}(\xi^\dagger D\eta+\eta^\dagger D\xi)-\hbar\,\mathrm{Im}[\xi^\dagger a(\partial^2-\nabla^2)(a\eta)+\eta^\dagger a(\partial^2-\nabla^2)(a\xi)] \tag{2.23}$$

using (2.4) and

$$\partial(a^2\omega\phi_q)+\nabla\cdot(a^2\omega A_q)=\hbar\,\mathrm{Im}[\xi^\dagger a(\partial^2-\nabla^2)(a\eta)+\eta^\dagger a(\partial^2-\nabla^2)(a\xi)]. \tag{2.24}$$

The fourth square bracket in equation (2.18) is given using equations (2.9) and (2.10) by

$$\phi_q(\partial S+\phi)+A_q\cdot(\nabla S-A)=\hbar\,\mathrm{Im}(\xi^\dagger D\eta+\eta^\dagger D\xi)/\omega. \tag{2.25}$$

Thus if we define

$$K\equiv H-\mathrm{i}E \tag{2.26}$$

the equations (2.18) may be written

$$a[(\partial S+\tilde{\phi})^2-(\nabla S-\tilde{A})^2-m^2-\phi_q^2+A_q^2]\binom{\xi}{\eta}+\hbar a\binom{K\cdot\sigma\xi}{K^*\cdot\sigma\eta}$$

$$-\hbar^2(\partial^2-\nabla^2)a\binom{\xi}{\eta}+\frac{\hbar^2\mathrm{i}}{\omega}\,\mathrm{Im}[\xi^\dagger(\partial^2-\nabla^2)(a\eta)+\eta^\dagger(\partial^2-\nabla^2)(a\xi)]\binom{\xi}{\eta} \tag{2.27}$$

$$=2\mathrm{i}\hbar a\left[D\binom{\xi}{\eta}-\frac{1}{\omega}(\xi^\dagger D\eta+\eta^\dagger D\xi)\binom{\xi}{\eta}\right].$$

Multiply this equation on the left by $(\eta^\dagger,\xi^\dagger)$. The result is real and is the Hamilton–Jacobi equation (Barut 1964, for example)

$$(\partial S+\tilde{\phi})^2-(\nabla S-\tilde{A})^2=(m-\Phi_q)^2 \tag{2.28}$$

for an electron in the presence of a four-vector potential $\tilde{\phi}$, $\tilde{\boldsymbol{A}}$ and a scalar potential Φ_q given by

$$a\omega(m-\Phi_q)^2=(m^2+\phi_q^2-\boldsymbol{A}_q^2)a\omega \\ +\hbar^2\,\mathrm{Re}[\eta^\dagger(\partial^2-\boldsymbol{\nabla}^2)a\xi+\xi^\dagger(\partial^2-\boldsymbol{\nabla}^2)a\eta]-2\hbar a\,\mathrm{Re}\,\eta^\dagger\boldsymbol{K}\cdot\boldsymbol{\sigma}\xi. \tag{2.29}$$

Using the HJ equation (2.28) we may rewrite equation (2.27) as

$$2\mathrm{i}\omega D\xi-2\mathrm{i}(\xi^\dagger D\eta+\eta^\dagger D\xi)\xi \\ =\omega\boldsymbol{K}\cdot\boldsymbol{\sigma}\xi-(\eta^\dagger\boldsymbol{K}\cdot\boldsymbol{\sigma}\xi+\xi^\dagger\boldsymbol{K}^*\cdot\boldsymbol{\sigma}\eta)\xi-\omega\xi_q+(\eta^\dagger\xi_q+\xi^\dagger\eta_q)\xi, \tag{2.30}$$

$$2\mathrm{i}\omega D\eta-2\mathrm{i}(\xi^\dagger D\eta+\eta^\dagger D\xi)\eta \\ =\omega\boldsymbol{K}^*\cdot\boldsymbol{\sigma}\eta-(\eta^\dagger\boldsymbol{K}\cdot\boldsymbol{\sigma}\xi+\xi^\dagger\boldsymbol{K}^*\cdot\boldsymbol{\sigma}\eta)\eta-\omega\eta_q+(\eta^\dagger\xi_q+\xi^\dagger\eta_q)\eta, \tag{2.31}$$

where

$$a\xi_q=\hbar(\partial^2-\boldsymbol{\nabla}^2)a\xi, \tag{2.32}$$

$$a\eta_q=\hbar(\partial^2-\boldsymbol{\nabla}^2)a\eta. \tag{2.33}$$

We will next show that these equations (2.30) and (2.31) are equivalent to a Frenkel equation for f given by

$$f\eta^\dagger\xi\equiv\eta^\dagger\boldsymbol{\sigma}\xi \tag{2.34}$$

together with an equation for θ, the phase of $\eta^\dagger\xi$, the real part of which is undetermined. To get the equation for the phase, multiply equation (2.30) on the left by $\eta^\dagger$, take the complex conjugate and subtract this from equation (2.31) multiplied on the left by $\xi^\dagger$. We get, using (2.4),

$$2\mathrm{i}\omega D\xi^\dagger\eta-2\mathrm{i}\xi^\dagger\eta D\omega=-(\xi^\dagger\eta+\eta^\dagger\xi)(\xi^\dagger\eta_q-\xi_q^\dagger\eta)+(\eta^\dagger\xi_q+\xi^\dagger\eta_q)\xi^\dagger\eta-(\xi_q^\dagger\eta+\eta_q^\dagger\xi)\xi^\dagger\eta, \tag{2.35}$$

$$2\mathrm{i}\eta^\dagger\xi D(\xi^\dagger\eta)-2\mathrm{i}\xi^\dagger\eta D(\eta^\dagger\xi)=\xi^\dagger\eta(\eta^\dagger\xi_q-\eta_q^\dagger\xi)-\eta^\dagger\xi(\xi^\dagger\eta_q-\xi_q^\dagger\eta), \tag{2.36}$$

$$D\theta=\mathrm{Re}[(\eta^\dagger\xi_q-\eta_q^\dagger\xi)/\eta^\dagger\xi]. \tag{2.37}$$

To get the Frenkel equation multiply equation (2.30) on the left by $\eta^\dagger(\boldsymbol{\sigma}-\boldsymbol{f})$,

$$2\mathrm{i}\eta^\dagger(\boldsymbol{\sigma}-\boldsymbol{f})D\xi=\eta^\dagger(\boldsymbol{\sigma}-\boldsymbol{f})\boldsymbol{K}\cdot\boldsymbol{\sigma}\xi-\eta^\dagger(\boldsymbol{\sigma}-\boldsymbol{f})\xi_q \tag{2.38}$$

$$=(\boldsymbol{K}-\boldsymbol{K}\cdot\boldsymbol{f}\boldsymbol{f})\eta^\dagger\xi-\mathrm{i}(\boldsymbol{K}+\boldsymbol{K}_q)\times\eta^\dagger\boldsymbol{\sigma}\xi, \tag{2.39}$$

where we have used equation (2.11) and the fact that $\eta^\dagger(\boldsymbol{\sigma}-\boldsymbol{f})\xi$ is zero and since (appendix 1) $\eta^\dagger(\boldsymbol{\sigma}-\boldsymbol{f})\xi_q$ is perpendicular to $\eta^\dagger\boldsymbol{\sigma}\xi$ we have written it in terms of $\boldsymbol{K}_q$ given by

$$\eta^\dagger(\boldsymbol{\sigma}-\boldsymbol{f})\xi_q=\mathrm{i}\boldsymbol{K}_q\times\eta^\dagger\boldsymbol{\sigma}\xi, \tag{2.40}$$

$$\boldsymbol{K}_q\cdot\eta^\dagger\boldsymbol{\sigma}\xi=0. \tag{2.41}$$

Similarly we get by multiplying equation (2.31) on the left by $\xi^\dagger(\boldsymbol{\sigma}-\boldsymbol{f}^*)$

$$2\mathrm{i}\xi^\dagger(\boldsymbol{\sigma}-\boldsymbol{f}^*)D\eta=(\boldsymbol{K}-\boldsymbol{K}\cdot\boldsymbol{f}\boldsymbol{f})^*\xi^\dagger\eta-\mathrm{i}(\boldsymbol{K}+\boldsymbol{K}_p)^*\times\xi^\dagger\boldsymbol{\sigma}\boldsymbol{\eta} \tag{2.42}$$

where $\boldsymbol{K}_p$ is defined by

$$\xi^\dagger(\boldsymbol{\sigma}-\boldsymbol{f}^*)\eta_q\equiv\mathrm{i}\boldsymbol{K}_p^*\times\xi^\dagger\boldsymbol{\sigma}\eta, \tag{2.43}$$

$$\boldsymbol{K}_p^*\cdot\xi^\dagger\boldsymbol{\sigma}\eta\equiv0. \tag{2.44}$$

Subtracting the complex conjugate of equation (2.42) from equation (2.39) we get

$$2D\eta^{\dagger}\boldsymbol{\sigma}\xi - 2fD\eta^{\dagger}\xi = -(2\boldsymbol{K} + \boldsymbol{K}_{\mathrm{q}} + \boldsymbol{K}_{\mathrm{p}}) \times \eta^{\dagger}\boldsymbol{\sigma}\xi \tag{2.45}$$

or

$$Df = -(\boldsymbol{K} + \tfrac{1}{2}\boldsymbol{K}_{\mathrm{q}} + \tfrac{1}{2}\boldsymbol{K}_{\mathrm{p}}) \times f. \tag{2.46}$$

We can interpret this equation as Frenkel's relativistic spin equation with an anomalous electromagnetic 'quantum' field.

The classical limit of this result was derived from the Dirac equation for the special cases of transverse and longitudinal constant electric and magnetic fields by Tolhock and de Groot (1951), but the attempt to generalise the fields seems to have been dropped when it was observed (Bargmann *et al* 1959) that the expectation value of the operator representing the spin necessarily follows the same time dependence as the relativistic classical equation of motion. These are uniquely the Frenkel equations (2.46) in the absence of the 'quantum' fields $\boldsymbol{K}_{\mathrm{q}}$ and $\boldsymbol{K}_{\mathrm{p}}$, which are equivalent to those of Bargmann *et al* (1959).

Equations (2.37) and (2.46) may also be written in terms of the substantive derivative

$$\tilde{D} = D + \phi_{\mathrm{q}}\partial + \boldsymbol{A}_{\mathrm{q}} \cdot \boldsymbol{\nabla} \tag{2.47}$$

corresponding to the world lines derived from the HJ equation (2.28). Replacing D by $\tilde{D}$ in (2.30) and *subsequent* equations and definitions may be compensated by replacing ξ_{q} and η_{q} by

$$\tilde{\xi}_{\mathrm{q}} = \xi_{\mathrm{q}} - 2\mathrm{i}(\phi_{\mathrm{q}}\partial\xi + \boldsymbol{A}_{\mathrm{q}} \cdot \boldsymbol{\nabla}\xi), \tag{2.48}$$

$$\tilde{\eta}_{\mathrm{q}} = \eta_{\mathrm{q}} - 2\mathrm{i}(\phi_{\mathrm{q}}\partial\eta + \boldsymbol{A}_{\mathrm{q}} \cdot \boldsymbol{\nabla}\eta). \tag{2.49}$$

Thus equations (2.37) and (2.44) become

$$\tilde{D}\theta = \mathrm{Re}[(\eta^{\dagger}\tilde{\xi}_{\mathrm{q}} - \tilde{\eta}_{\mathrm{q}}^{\dagger}\xi)/\eta^{\dagger}\xi], \tag{2.50}$$

$$\tilde{D}f = -(\boldsymbol{K} + \tfrac{1}{2}\tilde{\boldsymbol{K}}_{\mathrm{q}} + \tfrac{1}{2}\tilde{\boldsymbol{K}}_{\mathrm{q}}) \times f. \tag{2.51}$$

This Frenkel equation together with the HJ equation (2.28) and the continuity equation (2.16) are the hydrodynamic equations for a stream of non-interacting relativistic electrons in the presence of a quantum scalar and four-vector potential and anomalous electromagnetic field, affecting the spin, in addition to the classical fields. There is in the quantum theory additional phase information $\theta(x)$ which is constant along the paths of the electrons in the classical limit. The world lines of the electrons are determined by Hamilton's equations and the complex spin precesses about the complex magnetic field (classical plus 'quantum') that it meets on the way. These hydrodynamic equations are however coupled through their 'quantum' terms and are equivalent to the second-order Dirac equation (2.17).

3. The semiclassical approximation

If we were to use the HJ equation (2.28), the CE equation (2.16), the Frenkel equation (2.46) and the phase equation (2.37) to find the solution of the Dirac equation we would be faced with three difficulties.

First these equations determine $a(x)\omega(x)$, $S(x)$, $\theta(x)$ and $f(x)$ but leave $\omega(x)$, $\xi(x)$ and $\eta(x)$ undetermined. Knowledge of θ and f is *not* sufficient without some further constraint, for example

$$\xi^1 = 1, \tag{3.1}$$

to determine ξ and η since using equation (A1.1)

$$\xi^2 = \frac{1-f_z}{f_x + \mathrm{i}f_y}\xi^1 = \frac{f_x - \mathrm{i}f_y}{1+f_z}\xi^1, \tag{3.2}$$

$$2\eta_1^* = (1+f_z)\eta^+\xi/\xi^1, \tag{3.3}$$

$$2\eta_2^* = (f_x + \mathrm{i}f_y)\eta^+\xi/\xi^1. \tag{3.4}$$

Imposing the constraints (3.1) on equation (2.30), we get the following equation for ω:

$$-D\omega = \omega\alpha + \mathrm{Im}(\eta^+\xi_{\mathrm{q}} + \xi^+\eta_{\mathrm{q}}) \tag{3.5}$$

$$= \omega\alpha + a^{-2}[\partial(a^2\omega\phi_{\mathrm{q}}) + \boldsymbol{\nabla}\cdot(a^2\omega\boldsymbol{A}_{\mathrm{q}})] \tag{3.6}$$

where

$$\alpha(x) = \mathrm{Im}[K_z + (K_x + \mathrm{i}K_y)(f_x - \mathrm{i}f_y)/(1+f_z)]. \tag{3.7}$$

We have used equation (2.24) to derive equation (3.6).

The second difficulty is that unlike true hydrodynamic equations our HJ, CE, Frenkel and phase equations are coupled together through the quantum terms. They uncouple however in the semiclassical limit.

To zero order in $\hbar$, f satisfies Frenkel's spin equations so that it precesses about the $\boldsymbol{K}$ direction as the particle moves along a world line everywhere tangent to $\partial S + \phi$, $\boldsymbol{\nabla}S - \boldsymbol{A}$ determined by the classical HJ equation. Along this world line θ is constant but

$$\ln\omega = -\mathrm{Im}\int\left[K_z + \frac{(K_x + \mathrm{i}K_y)(f_x - \mathrm{i}f_y)}{1+f_z}\right]\mathrm{d}s \tag{3.8}$$

where the integral is taken along the world line and f has been determined along it by Frenkel's equation. Knowing θ and ω and hence $\eta^+\xi$, we can, using equations (3.2)–(3.4), determine ξ^2, η_1, η_2 from f and (3.1). These values of ξ and η allow us to calculate ϕ_{q}, $\boldsymbol{A}_{\mathrm{q}}$. a^2 is determined from the continuity equation (2.16).

Thus we find an approximate solution

$$\chi = \frac{a\,\mathrm{e}^{\mathrm{i}S/\hbar}}{4\cos\theta}\begin{pmatrix} 4\cos\theta \\ [4(f_x - \mathrm{i}f_y)/(1+f_z)]\cos\theta \\ (1+f_z^*)\,\mathrm{e}^{-\mathrm{i}\theta}\omega \\ (f_x^* - \mathrm{i}f_y^*)\,\mathrm{e}^{-\mathrm{i}\theta}\omega \end{pmatrix} \tag{3.9}$$

of the second-order Dirac equation (2.17).

The third difficulty is that the solution (3.9) of the second-order Dirac equation (2.17) is not necessarily a solution of the Dirac equation (2.2). However

$$\psi = \begin{pmatrix} m & \mathrm{i}\hbar - \phi + (\mathrm{i}\hbar\boldsymbol{\nabla} + \boldsymbol{A})\cdot\boldsymbol{\sigma} \\ \mathrm{i}\hbar - \phi - (\mathrm{i}\hbar\boldsymbol{\nabla} + \boldsymbol{A})\cdot\boldsymbol{\sigma} & m \end{pmatrix}$$

$$\times\frac{a\,\mathrm{e}^{\mathrm{i}S/\hbar}}{8m\cos\theta}\begin{pmatrix} 4\cos\theta \\ 4[(f_x - \mathrm{i}f_y)/(1+f_z)]\cos\theta \\ (1+f_z^*)\,\mathrm{e}^{-\mathrm{i}\theta}\omega \\ (f_x^* - \mathrm{i}f_y^*)\,\mathrm{e}^{-\mathrm{i}\theta}\omega \end{pmatrix} \tag{3.10}$$

is, since the second-order Dirac equation may be written

$$\begin{pmatrix} m & (\mathrm{i}\hbar-\phi)+(\mathrm{i}\hbar\nabla+\boldsymbol{A})\cdot\boldsymbol{\sigma} \\ (\mathrm{i}\hbar\partial-\phi)-(\mathrm{i}\hbar\nabla+\boldsymbol{A})\cdot\boldsymbol{\sigma} & m \end{pmatrix}$$

$$\times\begin{pmatrix} -m & (\mathrm{i}\hbar\partial-\phi)+(\mathrm{i}\hbar\nabla+\boldsymbol{A})\cdot\boldsymbol{\sigma} \\ \mathrm{i}\hbar\partial-\phi-(\mathrm{i}\hbar\nabla+\boldsymbol{A})\cdot\boldsymbol{\sigma} & -m \end{pmatrix}\chi$$

$$=\begin{pmatrix} -m & \mathrm{i}\hbar-\phi+(\mathrm{i}\hbar\nabla+\boldsymbol{A})\cdot\boldsymbol{\sigma} \\ \mathrm{i}\hbar\partial-\phi-(\mathrm{i}\hbar\nabla+\boldsymbol{A})\cdot\boldsymbol{\sigma} & -m \end{pmatrix}\psi=0. \tag{3.11}$$

The projection, equation (3.10), must be carried out on the solution given by the above approximation procedure in order to get an approximate solution to the Dirac equation.

4. The distorted wave Glauber approximation

In the distorted wave Glauber approximation (DWGA) we assume that we have a solution† ψ, equation (2.1), of the Dirac equation (2.2) for a vector potential ϕ, $\boldsymbol{A}$ and seek a solution of the second-order Dirac equation for a vector potential $\hat{\phi}$, $\hat{\boldsymbol{A}}$. That is, we wish to express $\hat{S}$, $\hat{a}$, $\hat{\theta}$, $\hat{\xi}$ and $\hat{\eta}$ in terms of S, a, θ, ξ and η. We will fix $\hat{\omega}$ and ω,

$$\hat{\omega}(\boldsymbol{x}, t)\equiv\omega(\boldsymbol{x}, t)\equiv 1, \tag{4.1}$$

as they are undetermined by the equations of § 2. This alternative to setting $\hat{\xi}^1=1$ leaves $\hat{S}$ unfixed for the present.

Although the Hamiltonian referred to in the HJ equation (2.28) is a scalar quantity, identically vanishing, nonetheless its Hamilton equations lead to the correct electron world lines for the given potentials. It will however in this section be convenient to reinterpret equation (2.28) by analogy with the Hamilton partial differential equation at fixed energy which has Hamilton's characteristic function, which depends on the energy, as its solution, rather than as the more general HJ equation which has the action, a function of time, as its solution. We will in this spirit rewrite equations (2.28) and (2.29) as

$$(\partial S+\tilde{\phi})^2-(\nabla S-\tilde{A})^2=m^2-V_{\mathrm{q}}, \tag{4.2}$$

$$-V_{\mathrm{q}}=\phi_{\mathrm{q}}^2-\boldsymbol{A}_{\mathrm{q}}^2+\hbar\,\mathrm{Re}(\eta^{\dagger}\tilde{\xi}_{\mathrm{q}}+\xi^{\dagger}\tilde{\eta}_{\mathrm{q}}-2\eta^{\dagger}\boldsymbol{K}\cdot\boldsymbol{\sigma}\xi), \tag{4.3}$$

where we have used equations (2.48) and (2.49). We interpret (4.2) as Hamilton's differential equation (and m^2 as the fixed 'energy'), which gives the path $\boldsymbol{x}(s)$, $t(s)$ of a particle in the four-dimensional 'space' $\boldsymbol{x}$, t with s an independent parameter analogous to time.

Our first ansatz

$$[\partial\hat{S}+\hat{\tilde{\phi}}]_{\boldsymbol{x}(s),t(s)}\equiv\dot{s}[\partial S+\tilde{\phi}]_{\boldsymbol{x}(s),t(s)}\equiv-m\dot{s}\frac{\mathrm{d}t}{\mathrm{d}s}, \tag{4.4}$$

$$[\nabla\hat{S}-\hat{\tilde{\boldsymbol{A}}}]_{\boldsymbol{x}(s),t(s)}\equiv\dot{s}[\nabla S-\tilde{\boldsymbol{A}}]_{\boldsymbol{x}(s),t(s)}\equiv m\dot{s}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s}, \tag{4.5}$$

† Any solution of the first-order Dirac equation is also a solution of the second-order Dirac equation, since the matrices on the left of equation (3.11) commute.

postulates that the paths are the same for both sets of potentials, merely re-parametrised by $s(\tau)$ so that the particles are travelling 'faster' for one set. Since $s(\tau)$ will in general differ from one path to the next, $\dot{s}$, the derivative of $s(\tau)$ with respect to τ, is a function of $\boldsymbol{x}$ and t. The continuity equation becomes

$$\partial[\hat{a}^2\dot{s}(\partial S+\tilde{\phi})]-\boldsymbol{\nabla}\cdot[\hat{a}^2\dot{s}(\boldsymbol{\nabla}S-\tilde{\boldsymbol{A}})]=0 \tag{4.6}$$

which together with the continuity equation (2.16) for the paths $\boldsymbol{x}(\tau)$, $t(\tau)$, implies

$$\tilde{D}\ln(\hat{a}^2\,\dot{s}/a^2)=0 \tag{4.7}$$

so that $\sqrt{\dot{s}}\hat{a}(x,t)/a(x,t)$ is a constant along each path,

$$\tilde{D}\dot{s}\equiv -m\left(\frac{\mathrm{d}t}{\mathrm{d}s}\partial+\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s}\boldsymbol{\nabla}\right)\dot{s}\equiv -m\frac{\mathrm{d}\tau}{\mathrm{d}s}\ddot{s}. \tag{4.8}$$

We may take the constant to be unity so that

$$\hat{a}(\boldsymbol{x},t)\equiv a(\boldsymbol{x},t)/\sqrt{\dot{s}}(\boldsymbol{x},t). \tag{4.9}$$

From here on we shall abbreviate functions $\xi(\boldsymbol{x},t)$ evaluated along the path $\boldsymbol{x}(s)$, $t(s)$ by $\xi(s)$.

Let us without loss of generality write

$$\hat{\xi}[s(\tau)]=T(\tau)\xi(\tau), \tag{4.10}$$

$$\hat{\eta}[s(\tau)]=W(\tau)\eta(\tau), \tag{4.11}$$

along each path. If the 2×2 complex matrices T, W satisfy

$$-2\mathrm{i}m\ (\mathrm{d}/\mathrm{d}\tau)T=\hat{\boldsymbol{K}}[s(\tau)]\cdot\boldsymbol{\sigma}T-T\boldsymbol{K}(\tau)\cdot\boldsymbol{\sigma}, \tag{4.12}$$

$$-2\mathrm{i}m\ (\mathrm{d}/\mathrm{d}\tau)W=\hat{\boldsymbol{K}}^*[s(\tau)]\cdot\boldsymbol{\sigma}W-W\boldsymbol{K}^*(\tau)\cdot\boldsymbol{\sigma}, \tag{4.13}$$

which imply

$$-2\mathrm{i}m\ (\mathrm{d}/\mathrm{d}\tau)\hat{\xi}+2\mathrm{i}mT\ (\mathrm{d}/\mathrm{d}\tau)\xi=\hat{\boldsymbol{K}}\cdot\boldsymbol{\sigma}\hat{\xi}-T\boldsymbol{K}\cdot\boldsymbol{\sigma}\xi, \tag{4.14}$$

$$-2\mathrm{i}m\ (\mathrm{d}/\mathrm{d}\tau)\hat{\eta}+2\mathrm{i}mW\ (\mathrm{d}/\mathrm{d}\tau)\eta=\hat{\boldsymbol{K}}^*\cdot\boldsymbol{\sigma}\hat{\eta}-W\boldsymbol{K}^*\cdot\boldsymbol{\sigma}\eta, \tag{4.15}$$

then the equations for $\hat{\xi}$ and $\hat{\eta}$

$$2\mathrm{i}\hat{\tilde{D}}\hat{\xi}-2\mathrm{i}(\hat{\xi}^{\dagger}\hat{\tilde{D}}\hat{\eta}+\hat{\eta}^{\dagger}\hat{\tilde{D}}\hat{\xi})\hat{\xi}$$
$$=\hat{\boldsymbol{K}}\cdot\boldsymbol{\sigma}\hat{\xi}-(2\,\mathrm{Re}\hat{\eta}^{\dagger}\hat{\boldsymbol{K}}\cdot\boldsymbol{\sigma}\hat{\xi})\hat{\xi}-\hat{\tilde{\xi}}_{\mathrm{q}}+(\hat{\eta}^{\dagger}\hat{\tilde{\xi}}_{\mathrm{q}}+\hat{\xi}^{\dagger}\hat{\tilde{\eta}}_{\mathrm{q}})\hat{\xi}, \tag{4.16}$$

$$2\mathrm{i}\hat{\tilde{D}}\hat{\eta}-2\mathrm{i}(\hat{\xi}^{\dagger}\hat{\tilde{D}}\hat{\eta}+\hat{\eta}^{\dagger}\hat{\tilde{D}}\hat{\xi})\hat{\eta}$$
$$=\hat{\boldsymbol{K}}^*\cdot\boldsymbol{\sigma}\hat{\eta}-(2\,\mathrm{Re}\,\hat{\eta}^{\dagger}\hat{\boldsymbol{K}}\cdot\boldsymbol{\sigma}\hat{\xi})\hat{\eta}-\hat{\tilde{\eta}}_{\mathrm{q}}+(\hat{\eta}^{\dagger}\hat{\tilde{\xi}}_{\mathrm{q}}+\hat{\xi}^{\dagger}\hat{\tilde{\eta}}_{\mathrm{q}})\hat{\eta}, \tag{4.17}$$

are satisfied, using equations (2.30) and (2.31), provided the approximation

$$\hat{\tilde{\xi}}_{\mathrm{q}}[s(\tau)]=T(\tau)\tilde{\xi}_{\mathrm{q}}(\tau), \tag{4.18}$$

$$\hat{\tilde{\eta}}_{\mathrm{q}}[s(\tau)]=W(\tau)\tilde{\eta}_{\mathrm{q}}(\tau), \tag{4.19}$$

is valid, which depends on how $\hat{a}$, T and W vary from one path to another.

From equations (4.12) and (4.13) we find that

$$2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}(W^{\dagger}T)=2\mathrm{i}\left[W^{\dagger}\frac{\mathrm{d}}{\mathrm{d}\tau}T+\left(\frac{\mathrm{d}}{\mathrm{d}\tau}W^{\dagger}\right)T\right]=-\frac{1}{m}[\boldsymbol{K}(\tau)\cdot\boldsymbol{\sigma},W^{\dagger}T], \tag{4.20}$$

thus if $W^{\dagger}T$ is the unit matrix for τ zero it is so for all τ, thus $W^{\dagger}$ is the inverse of T

if we assume that asymptotically

$$\hat{\xi}[s(0)]=\xi(0), \tag{4.21}$$

$$\hat{\eta}[s(0)]=\eta(0). \tag{4.22}$$

Then

$$\hat{\eta}^{\dagger}[s(\tau)]\hat{\xi}[s(\tau)]=\eta^{\dagger}(\tau)\xi(\tau) \tag{4.23}$$

so that

$$\hat{\theta}[s(\tau)]=\theta(\tau). \tag{4.24}$$

The expression for $\hat{S}[s(\tau)]$ may be found by observing that the solution of the HJ equations

$$(\partial\hat{S}+\hat{\boldsymbol{\phi}})^2-(\boldsymbol{\nabla}S-\hat{\boldsymbol{A}})^2=m^2-\hat{V}_{\mathrm{q}} \tag{4.25}$$

and (4.2) may be written using (4.4) and (4.5),

$$\hat{S}(s)=-\int^{s}\hat{\boldsymbol{\phi}}(s)\frac{\mathrm{d}t(s)}{\mathrm{d}s}\,\mathrm{d}s+\int^{s}\hat{\boldsymbol{A}}(s)\cdot\frac{\mathrm{d}\boldsymbol{x}(s)}{\mathrm{d}s}\,\mathrm{d}s-\frac{1}{m}\int^{s}[m^2-\hat{V}_{\mathrm{q}}(s)]\frac{\mathrm{d}\tau}{\mathrm{d}s}\,\mathrm{d}s, \tag{4.26}$$

$$S(\tau)=-\int^{\tau}\tilde{\boldsymbol{\phi}}(s)\frac{\mathrm{d}t(s)}{\mathrm{d}s}\,\mathrm{d}s+\int^{\tau}\tilde{\boldsymbol{A}}(s)\cdot\frac{\mathrm{d}\boldsymbol{x}(s)}{\mathrm{d}s}\,\mathrm{d}s-\frac{1}{m}\int^{\tau}[m^2-V_{\mathrm{q}}(s)]\,\mathrm{d}s. \tag{4.27}$$

Thus

$$\begin{aligned}
\hat{S}[s(\tau)]-S(\tau)+\int^{\tau}\Big\{\hat{\boldsymbol{\phi}}[s(\tau)]\frac{\mathrm{d}t[s(\tau)]}{\mathrm{d}\tau}-\boldsymbol{\phi}(\tau)\frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau}\\
-\hat{\boldsymbol{A}}[s(\tau)]\cdot\frac{\mathrm{d}\boldsymbol{x}[s(\tau)]}{\mathrm{d}\tau}+\boldsymbol{A}(\tau)\frac{\mathrm{d}\boldsymbol{x}(\tau)}{\mathrm{d}\tau}\Big\}\,\mathrm{d}\tau\\
=\int^{\tau}\Big(-\hat{\phi}_{\mathrm{q}}[s(\tau)]\frac{\mathrm{d}t[s(\tau)]}{\mathrm{d}\tau}+\phi_{\mathrm{q}}(\tau)\frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau}+\hat{\boldsymbol{A}}_{\mathrm{q}}[s(\tau)]\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau}[s(\tau)]\\
-\boldsymbol{A}_{\mathrm{q}}(\tau)\cdot\frac{\mathrm{d}\boldsymbol{x}(\tau)}{\mathrm{d}\tau}+\frac{1}{m}\hat{V}_{\mathrm{q}}[s(\tau)]-\frac{1}{m}V_{\mathrm{q}}(\tau)\Big)\,\mathrm{d}\tau
\end{aligned} \tag{4.28}$$

$$=-\frac{1}{m}\int^{\tau}[(\hat{\phi}_{\mathrm{q}}^2-\hat{\boldsymbol{A}}_{\mathrm{q}}^2)_{s(\tau)}-(\phi_{\mathrm{q}}^2-\boldsymbol{A}_{\mathrm{q}}^2)_{\tau}]\,\mathrm{d}\tau \tag{4.29}$$

using equations (4.18), (4.19) and the fact that equations (4.14), (4.15) imply

$$\begin{aligned}
-(m/\hbar)\beta(\tau)&\equiv\mathrm{Re}[(\hat{\eta}^{\dagger}\hat{\boldsymbol{K}}\cdot\boldsymbol{\sigma}\hat{\xi})_{s(\tau)}-(\eta^{\dagger}\boldsymbol{K}\cdot\boldsymbol{\sigma}\xi)_{\tau}]\\
&=\mathrm{Re}\,\eta^{\dagger}(\tau)\{T^{-1}\hat{\boldsymbol{K}}[s(\tau)]\cdot\boldsymbol{\sigma}T-\boldsymbol{K}(\tau)\cdot\boldsymbol{\sigma}\}\xi(\tau)
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
&=m\,\mathrm{Im}\Big(\hat{\eta}^{\dagger}[s(\tau)]\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\xi}[s(\tau)]+\hat{\xi}^{\dagger}[s(\tau)]\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\eta}[s(\tau)]\\
&\quad-\eta^{\dagger}(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\xi(\tau)-\xi^{\dagger}(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\eta(\tau)\Big)
\end{aligned} \tag{4.31}$$

$$=\frac{m}{\hbar}\Big(\hat{\phi}_{\mathrm{q}}[s(\tau)]\frac{\mathrm{d}t[s(\tau)]}{\mathrm{d}\tau}-\hat{\boldsymbol{A}}_{\mathrm{q}}[s(\tau)]\cdot\frac{\mathrm{d}\boldsymbol{x}[s(\tau)]}{\mathrm{d}\tau}-\phi_{\mathrm{q}}(\tau)\frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau}+\boldsymbol{A}_{\mathrm{q}}(\tau)\cdot\frac{\mathrm{d}\boldsymbol{x}(\tau)}{\mathrm{d}\tau}\Big) \tag{4.32}$$

using the definitions (2.9) and (2.10).

It is shown in appendix 2 that approximating the 'vector' $\hat{A}_q^\mu - A_q^\mu = (\hat{\phi}_q[s(\tau)] - \phi_q(\tau), \hat{\boldsymbol{A}}_q[s(\tau)] - \boldsymbol{A}_q(\tau))$ by its component along the 'velocity' $dx^\mu/ds = (dt/ds, d\boldsymbol{x}/ds)$ is tantamount to assuming that neither the 'velocity' nor $s(\tau)$, $T(\tau)$ differ much between adjacent points on different paths. The component along the velocity follows from equation (4.32)

$$[\hat{\boldsymbol{A}}_q[s(\tau)] - \boldsymbol{A}_q(\tau)] \cdot \frac{d\boldsymbol{x}[s(\tau)]}{d\tau} - [\hat{\phi}_q[s(\tau)] - \phi_q(\tau)] \frac{dt[s(\tau)]}{d\tau}$$

$$= \boldsymbol{A}_q(\tau) \cdot \frac{d}{d\tau}[\boldsymbol{x}(\tau) - \boldsymbol{x}[s(\tau)]] - \phi_q(\tau) \frac{d}{d\tau}[t(\tau) - t[s(\tau)]] + \beta(\tau). \tag{4.33}$$

Thus the argument of the integral on the right of equation (4.29) may be written in this approximation

$$(\hat{\phi}_q^2 - \hat{\boldsymbol{A}}_q^2)_{s(\tau)} - (\phi_q^2 - \boldsymbol{A}_q^2)_\tau$$

$$= (\hat{A}_q^\mu[s(\tau)] - A_q^\mu(\tau))(\hat{A}_{q\mu}[s(\tau)] + A_{q\mu}(\tau)) \tag{4.34}$$

$$= \frac{-\{A_q(\tau) \cdot [\dot{x}[s(\tau)] - \dot{x}(\tau)] + \beta(\tau)\}\{A_q(\tau) \cdot [\dot{x}[s(\tau)] + \dot{x}(\tau)] - \beta(\tau)\}}{\dot{x}[s(\tau)] \cdot \dot{x}[s(\tau)]} \tag{4.35}$$

where the dot product is now of four-dimensional 'vectors' and $\dot{x}[s(\tau)]$ denotes the derivative of $x[s(\tau)]$ with respect to τ.

We are now in a position to derive an equation for $s(\tau)$. The HJ equations (4.2) and (4.25) taken together with equations (4.4) and (4.5) and (4.35) give

$$[\dot{x}[s(\tau)] - \dot{x}(\tau)] \cdot [\dot{x}[s(\tau)] + \dot{x}(\tau)]m^2$$

$$= -\hat{V}_q[s(\tau)] + V_q(\tau)$$

$$= \frac{\{A_q(\tau) \cdot [\dot{x}[s(\tau)] - \dot{x}(\tau)] + \beta\}\{-A_q(\tau) \cdot [\dot{x}[s(\tau)] + \dot{x}(\tau)] + \beta\}}{\dot{x}[s(\tau)] \cdot \dot{x}[s(\tau)]} + 2\beta m \tag{4.36}$$

which since $x(s)$ and $A_q(\tau)$ are known is a first-order nonlinear differential equation for $s(\tau)$. Unfortunately $\beta(\tau)$ depends on $T(\tau)$ unless this commutes with $\hat{\boldsymbol{K}}[s(\tau)] \cdot \boldsymbol{\sigma}$, as we shall assume for our first approximation.

The first step in the distorted wave Glauber approximation is to solve the first-order differential equation (4.36) for $s(\tau)$. Using this $s(\tau)$ we then solve the first-order spin equations (4.12) (and use this $T(\tau)$ to derive an improved $s(\tau)$ from equation (4.36)) or (4.14) and (4.15) and evaluate the integral of the expression on the right of (4.35) to get $\hat{S}$ from (4.29). Remember that $\hat{\omega}$ is unity and $\hat{a}(\boldsymbol{x}, t)$ is given by equations (4.9) so that we have the solution

$$\hat{\chi}(\boldsymbol{x}, t) = \hat{a}(\boldsymbol{x}, t)\, e^{i\hat{S}(\boldsymbol{x},t)/\hbar} \begin{pmatrix} \hat{\xi}(\boldsymbol{x}, t) \\ \hat{\eta}(\boldsymbol{x}, t) \end{pmatrix} \tag{4.37}$$

of the perturbed second-order Dirac equation. To get a solution of the perturbed first-order equation we must take

$$\hat{\psi}(\boldsymbol{x}, t) = \begin{pmatrix} m & i\hbar\partial - \hat{\phi} + (i\hbar\boldsymbol{\nabla} + \hat{\boldsymbol{A}}) \cdot \boldsymbol{\sigma} \\ i\hbar\partial - \hat{\phi} - (i\hbar\boldsymbol{\nabla} + \hat{\boldsymbol{A}}) \cdot \boldsymbol{\sigma} & m \end{pmatrix} \hat{a}(\boldsymbol{x}, t)\, e^{i\hat{S}(\boldsymbol{x},t)/\hbar} \begin{pmatrix} \hat{\xi}(\boldsymbol{x}, t) \\ \hat{\eta}(\boldsymbol{x}, t) \end{pmatrix}. \tag{4.38}$$

5. Conclusions

The Dirac equation in the presence of the potential field $(\phi, \boldsymbol{A})$ is equivalent to the following hydrodynamic equations coupled together by the presence of the 'quantum' four-vector potential field $(\phi_q, \boldsymbol{A}_q)$ defined by (2.9) and (2.10) and the scalar field Φ_q defined by (2.29), together with an anomalous 'quantum' electromagnetic field $\boldsymbol{H}_{\text{quantum}} + \mathrm{i}\boldsymbol{E}_{\text{quantum}} = \frac{1}{2}(\boldsymbol{K}_q + \boldsymbol{K}_p)$ given by (2.40), (2.41), (2.43) and (2.44) rather than by differential operations on $(\phi_q, \boldsymbol{A}_q)$ similar to (2.19) and (2.20) which define $\boldsymbol{H}$, $\boldsymbol{E}$: (1) the HJ equation (2.28) for an electron in the presence of a four-vector potential $(\phi + \phi_q, \boldsymbol{A} + \boldsymbol{A}_q)$ and a scalar potential Φ_q; (2) the Frenkel relativistic equation (2.51) for the spin of an electron moving through space in accordance with the HJ equation with a magnetic field $\boldsymbol{H} + \boldsymbol{H}_{\text{quantum}}$, and electric field $\boldsymbol{E} + \boldsymbol{E}_{\text{quantum}}$ acting on the spin; (3) the continuity equation (2.15) governing the four-vector current $(\rho, \boldsymbol{j})$ given by (2.6) and (2.12) of electrons moving in accordance with the HJ equation and not interacting with one another; (4) an equation (2.50) for the additional phase information θ which appears only in the relativistic quantum theory. Given any breakdown of any given solution of the Dirac equation into the components S, a, ξ, η, these components satisfy the coupled hydrodynamic equations of § 2. One could simplify the equations by choosing S to be a solution of the classical HJ equation and a^2 to be a solution of the corresponding classical continuity equation. In this paper however it is assumed that the solutions of the classical equations are not given. So the paths $t(\tau)$, $\boldsymbol{x}(\tau)$ are determined by S, $\phi + \phi_q$ and $\boldsymbol{A} + \boldsymbol{A}_q$.

In the classical limit $\hbar = 0$ the equations decouple and θ is trivial, giving the classical hydrodynamic equations for a stream of relativistic electrons which interact with the potential ϕ, $\boldsymbol{A}$ but not with one another, the spin being governed by Frenkel's equation in an electromagnetic field derived from this potential. The semiclassical approximation requires the solution of a HJ equation and a Frenkel equation. It is presented here so that the Glauber-type approximation may be compared with it and ordinary perturbation theory.

The distorted wave Glauber approximation presented in § 4 requires neither $\hbar$ to be negligible nor the perturbation $\phi + \phi_q$, $\boldsymbol{A} + \boldsymbol{A}_q$ to be small. There is however a smoothness condition on the perturbation expressed by the requirement that the ansatz (4.4), (4.5) be compatible with the HJ equations (4.2) and (4.25). That is, the electron is moving along the unperturbed 'quantum' path sufficiently fast and the perturbation potential is so smooth that the force due to it is too small to drive the electron far from this path, but merely speeds it up along the path. The advantage of the method is that the classical HJ equation does not have to be solved. The paths $\boldsymbol{x}(s)$, $t(s)$ used throughout are read off from the solution of the unperturbed Dirac equation and do not require the solution of the unperturbed HJ equation. We are then left with first-order differential equations (4.36) for the speeding up function $s(\tau)$, (4.12) for the spin matrix $T(\tau)$ and the evaluation of an integral (4.29) along the given paths to determine the phase S. The method works well when the unperturbed potential is that of an electromagnetic plane wave and the perturbation does not vary sufficiently in space and time to drive a fast moving electron far from the classical path that the electron has in the presence of the plane wave alone. It also seems to work well with the Coulomb potential as the unperturbed system and perturbations which do not vary greatly in space or time. To compare the DWGA with the semiclassical approximation and ordinary perturbation theory, these should be applied where the exact solution may be found both for the unperturbed and perturbed potentials. In the only such cases I know either the DWGA gives the exact answer or it reduces to the ordinary Glauber approximation.

Appendix 1. *Lemma*: $\eta^\dagger(\boldsymbol{\sigma}-\boldsymbol{f})\xi_q$ *is perpendicular to* $\eta^\dagger\boldsymbol{\sigma}\xi$ *(used to derive (2.39)).*

For two spinors ξ and $\hat{\xi}$ the dot product of the vector $\eta^\dagger\boldsymbol{\sigma}\xi$ given by

$$\eta^\dagger\sigma_x\xi,\ \eta^\dagger\sigma_y\xi,\ \eta^\dagger\sigma_z\xi = \eta_1^*\xi^2+\eta_2^*\xi^1,\ \mathrm{i}(\eta_1^*\xi^2-\eta_2^*\xi^1),\ \eta_1^*\xi^1-\eta_2^*\xi^2 \tag{A1.1}$$

with the vector $\eta^\dagger\boldsymbol{\sigma}\hat{\xi}$ is

$$\eta^\dagger\boldsymbol{\sigma}\xi\cdot\eta^\dagger\boldsymbol{\sigma}\hat{\xi} = \eta_1^*\eta_2^*(\xi^2\hat{\xi}^1+\xi^1\hat{\xi}^2)+\eta_1^*\eta_1^*\xi^1\hat{\xi}^1+\eta_2^*\eta_2^*\xi^2\hat{\xi}^2 \tag{A1.2}$$

$$= (\eta_1^*\xi^1+\eta_2^*\xi^2)(\eta_1^*\hat{\xi}^1+\eta_2^*\xi^2)$$

$$= \eta^\dagger\xi \quad \eta^\dagger\hat{\xi}. \tag{A1.3}$$

Thus if either ξ or $\hat{\xi}$ is orthogonal to η the vectors are perpendicular. Using equation (A1.3) the complex vector f defined by (2.34) satisfies

$$f\cdot f=1. \tag{A1.4}$$

If one of ξ, $\hat{\xi}$ is orthogonal to η and the other is not then any spinor ξ_q may be written as a superposition of them,

$$\xi_q=\alpha\xi+\beta\hat{\xi}; \tag{A1.5}$$

then

$$\eta^\dagger(\boldsymbol{\sigma}-f)\xi_q=\beta\eta^\dagger\boldsymbol{\sigma}\hat{\xi} \tag{A1.6}$$

is perpendicular to $\eta^\dagger\boldsymbol{\sigma}\xi$ and so may be written in terms of $\boldsymbol{K}_q$ given by

$$\eta^\dagger(\boldsymbol{\sigma}-f)\xi_q=\mathrm{i}\boldsymbol{K}_q\times\eta^\dagger\boldsymbol{\sigma}\xi, \tag{A1.7}$$

$$\boldsymbol{K}_q\cdot\eta^\dagger\boldsymbol{\sigma}\xi=0, \tag{A1.8}$$

thus defining a quantum electromagnetic field $\boldsymbol{H}_q$, $\boldsymbol{E}_q$

$$\boldsymbol{K}_q=\boldsymbol{H}_q-\mathrm{i}\boldsymbol{E}_q \tag{A1.9}$$

perpendicular to f and of norm $\mathrm{i}\beta\eta^\dagger\hat{\xi}/\eta^\dagger\xi$ because using (A1.3)

$$-\beta^2(\eta^\dagger\hat{\xi})^2=-\beta^2\eta^\dagger\boldsymbol{\sigma}\hat{\xi}\cdot\eta^\dagger\boldsymbol{\sigma}\hat{\xi} \tag{A.10}$$

$$=(\boldsymbol{K}_q\times\eta^\dagger\boldsymbol{\sigma}\xi)\cdot(\boldsymbol{K}_q\times\eta^\dagger\boldsymbol{\sigma}\xi) \tag{A1.11}$$

$$=\boldsymbol{K}_q\cdot\boldsymbol{K}_q(\eta^\dagger\boldsymbol{\sigma}\xi\cdot\eta^\dagger\boldsymbol{\sigma}\xi)-(\boldsymbol{K}_q\cdot\eta^\dagger\boldsymbol{\sigma}\xi)^2 \tag{A1.12}$$

$$=\boldsymbol{K}_q\cdot\boldsymbol{K}_q(\eta^\dagger\xi)^2. \tag{A1.13}$$

Except for isolated points ξ will not be orthogonal to η, so the norm of $\boldsymbol{K}_q$ is zero, and therefore

$$\boldsymbol{H}_q\cdot\boldsymbol{H}_q=\boldsymbol{E}_q\cdot\boldsymbol{E}_q, \tag{A1.14}$$

$$\boldsymbol{H}_q\cdot\boldsymbol{E}_q=0, \tag{A1.15}$$

the magnetic and electrostatic energy of the 'quantum' field are equal and the electric and magnetic fields are perpendicular to one another and to the spin direction.

Appendix 2. *(The approximation used in equation (4.35).)*

Let the four 'vectors' $x(s)$ and $x_2(s)$ denote the paths which run back from adjacent points $x[s(\tau)]$ and

$$x_2[s(\tau)] = x[s(\tau)] + \delta x; \tag{A2.1}$$

then

$$\hat{\xi}[s(\tau)]^\dagger[\hat{\eta}[s(\tau)] - \hat{\eta}[x_2(\tau)]] = \xi(\tau)^\dagger[\eta(\tau) - \eta[x_2(\tau)]] \tag{A2.2}$$

if $T(\tau)$ and $s(\tau)$ are the same for both paths. Thus in the limit of δx small

$$\hat{A}_q[s(\tau)] \cdot [x_2[s(\tau)] - x[s(\tau)]] = A_q(\tau) \cdot [x_2(\tau) - x(\tau)] \tag{A2.3}$$

so

$$[\hat{A}_q[s(\tau)] - A_q(\tau)] \cdot [x_2[s(\tau)] - x[s(\tau)]]$$
$$= A_q(\tau) \cdot \{x_2(\tau) - x_2[s(\tau)] - x(\tau) + x[s(\tau)]\}. \tag{A2.4}$$

But the braces on the right of equation (A2.4) is proportional to the difference between the average velocity on each path which is approximately zero if δx is perpendicular to the paths.

We note that if we allowed $T(\tau)$ to differ from path to adjacent path by a phase this could be accommodated by a change in $\hat{S}(x, t)$ allowed for in the discussion succeeding equation (4.1).

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